

# Exact radial solution in 2+1 gravity with a real scalar field

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## Abstract

We present an exact solution to 2+1 dimensional gravity plus a self-interacting real scalar field. The spatial part of the metric of the solution is flat but the temporal part behaves like asymptotic Anti-de Sitter space-time. The scalar field has logarithmic behavior as one would expect for a massless scalar field in flat space-time. The metric of the solution has no horizons and is thus not a black hole space-time. The solution can be most closely likened to gravitating scalar field solutions in 3 + 1 dimensions but with certain oddities connected with the 2 + 1 dimensional character of the space-time. One such odd feature is that the energy density,  $T_{00}$  is zero. The space-time is curved entirely due to a non-zero radial pressure,  $T_{rr}$ . The present solution is unique to 2 + 1 dimensions; it does not carry over to 3 + 1 dimensions.

Keywords: 2+1 dimensional gravity, exact solution, BTZ black hole,  
self-interacting scalar field

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# 1 Introduction

The study of general relativity in dimensions larger than the  $3 + 1$  that are observed has a history almost as long as general relativity itself. Already in [1] a  $4 + 1$  dimensional space-time was investigated in an attempt to unify the gravitational and the electromagnetic interactions. In general, Kaluza-Klein theories [2] and string theory [3] work with space-time dimensions greater than  $3 + 1$ . The study of general relativity in lower dimensions – specifically  $2 + 1$  dimensions – has a more recent history. Starting in the mid-1980s several researchers launched investigations into  $2 + 1$  gravity [4], [5], [6], [7], and [8]. A recent review of  $2 + 1$  gravity can be found in [9]. The motivation was to understand gravity in the simpler context of  $2 + 1$  dimensions in the hope that this would help shed some light on the more realistic but more complex  $3 + 1$  dimensional case – in particular in regard to the question of quantizing gravity.

There are some well known differences [4], [5] between gravity in  $2 + 1$  and  $3 + 1$  dimensions. First, in the absence of sources,  $2 + 1$  space-times are locally flat. Related to this is the fact that the Weyl tensor vanishes in  $2 + 1$  dimensional gravity. Second, in  $2 + 1$  dimensional gravity there are no gravitational waves. Third, in  $2 + 1$  dimensions there exists no adequate Newtonian limit for the Einstein field equation [5]. Finally, many geometric and topological properties of spaces behave completely different in even dimensions in comparison to an odd number of dimensions. For example [10], antipodal identification within the standard  $n$ -sphere  $S^n$  leads to an orientable factor space for odd values of  $n$  only.

One might surmise that since  $2 + 1$  space-time is flat outside the presence of sources that a point source in  $2 + 1$  dimensions would have no effect except exactly at the location of the source. However, in [4] and [5], single and multiple point sources were studied and it was found that a point source introduced a conical singularity in the space-time. Thus although  $2 + 1$  space-time was locally flat it would be globally conical with the deficit angle depending on the mass of the point source. Light rays in the presence of a  $2 + 1$  dimensional point mass, although moving locally in flat space-time, could be “lensed” due to the global conical structure of the space-time.

Given the requirement that the local presence of a mass-energy source was required to curve space-time it was a surprise that it was possible to construct

a black hole<sup>2</sup> space-time [11] (a review of  $2+1$  dimensional BTZ black holes is given in [12]). This was accomplished by considering a point mass source in a space-time with a negative cosmological constant,  $\Lambda < 0$  – i.e. Anti-de Sitter (AdS) space-time. Interpreting this negative cosmological constant as an energy-momentum tensor  $T_{\mu\nu}^{AdS} \propto \Lambda g_{\mu\nu}$  would imply a negative energy density  $\rho^{AdS} < 0$ . We point this out to emphasise that sources in  $2+1$  dimensions can be odd or anti-intuitive from a  $3+1$  perspective. The exact solution we give below fits into this description since we will find that the curvature of the  $2+1$ -dimensional space-time comes entirely from a non-zero, radial pressure.

In this paper we study the system of  $2+1$  gravity plus a self-interacting, real scalar field and give a simple, exact solution to this system. To our knowledge not much work has been done in investigating gravitating scalar fields in  $2+1$  dimensions. There have been numerous studies of gravitating scalar fields in  $3+1$  dimensions. For a complex scalar field in  $3+1$  dimensions [13] both without as well as with self-interactions [14] it was shown that one could find boson “star” solutions – localized, spherically symmetric configurations of the scalar field. Real scalar fields in  $3+1$  dimensions were investigated in [15], [16], and [17]. In the case of a real scalar field it was not possible to find a non-singular solution *except* in the case when one allowed the scalar field to be a phantom field [15], i.e. the kinetic energy term for the scalar field had a negative sign. However in this case of a phantom, real scalar field, the topology of the solution became that of a wormhole. On another level, non-singular solutions in  $3+1$  dimensions with a massive real

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<sup>2</sup>In [5] it was explicitly stated (see p. 759): “Black holes do not exist in a 3-dimensional space-time.” However, one should add that there is nothing wrong with this statement in [5] given the context of the paper since the definition of a black hole used in [5] is stronger than the definition used for a BTZ black hole. For the usual  $3+1$  Schwarzschild black hole, as in many other cases, the horizon is defined by global considerations. However the horizon in these cases is located at exactly the same hypersurface as a Killing horizon, i.e. a hypersurface where one of the Killing vectors changes its character from space-like to time-like. This latter condition is defined locally, and is much easier to verify. In summary for many space-times, like the  $3+1$  dimensional Schwarzschild metric, the definition of the black hole via the Killing horizon is equivalent to the horizon defined via global considerations since these two definitions of horizon coincide in these cases. However, in the case of the  $2+1$  dimensional BTZ space-time the situation is different since this metric does not possess a Killing horizon. In short: The BTZ black hole is locally static at all of its points, whereas the Schwarzschild black hole is locally static outside the horizon only.

scalar field as source have been shown to exist in Einstein's theory within the set of closed Friedmann universe models, see e.g. [18].

There have been some studies of gravitating scalar fields in  $2 + 1$  dimensions but almost universally connected with time dependent situations such as collapse to a BTZ black hole [19]. As far as we have found there has only been one study of  $2 + 1$  static solutions [20] similar to those investigated in  $(3 + 1)$ , see [15], [16], and [17]. This article presents one such static solution of a gravitating scalar field in  $2 + 1$ . The exact solution presented here can be thought of as a localized particle-like solution in  $2 + 1$  dimensions since the curvature scalars – the Ricci scalar  $R_\alpha^\alpha$  and Ricci tensor squared,  $R^{\alpha\beta}R_{\alpha\beta}$  – behave like  $1/r^2$  and  $1/r^4$  respectively. The singularity in these scalars at  $r = 0$  can be thought of as the location of the scalar “particle”. Additionally the scalar field behaves like  $\ln(r)$  which is typical of a massless field in  $2 + 1$  dimensions in the absence of gravity with the source located at  $r = 0$ . While the spatial part of our metric is flat (i.e.  $dr^2 + r^2d\theta^2$ ) the temporal part of our metric behaves like the asymptotic form of AdS space-time (i.e. the temporal part of the metric is  $Kr^2dt^2$  which is the same as the temporal part of AdS space-time in the limit  $r \rightarrow \infty$ ). This can be contrasted with the BTZ solution which is asymptotically AdS space-time in both the time and radial coordinate.

## 2 Metric and solution

We begin by writing Einstein's equation for  $2 + 1$  dimensions

$$G_{\mu\nu} = \kappa T_{\mu\nu} , \quad (2.1)$$

where  $\kappa = 8\pi G_3$  is the  $2 + 1$  dimensional gravitational coupling and  $G_3$  is Newton's constant in  $2 + 1$  dimensions;  $\kappa$  has dimensions of inverse mass in units where  $c = 1$ .<sup>3</sup> We use the convention of Landau and Lifshitz [21]. The metric ansatz we take as

$$ds^2 = A(r)dt^2 - dr^2 - r^2d\theta^2 , \quad (2.2)$$

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<sup>3</sup>In [5], the following argument is given: In  $n$ -dimensional space-times and in units where  $c = 1$ , the dimension of  $\kappa$  is  $\text{length}^{n-3}/\text{mass}$ , this explains the special role of  $n = 3$ .

We first calculate  $G_{\mu\nu}$  for eq. (2.2) to get the left hand side of the 2 + 1 Einstein's equation (2.1)

$$G_{rr} = \frac{A'(r)}{2rA(r)} \quad ; \quad G_{\theta\theta} = \frac{r^2}{4A^2(r)} \left[ -(A'(r))^2 + 2A(r)A''(r) \right] . \quad (2.3)$$

All other components of  $G_{\mu\nu}$  are zero. One oddity is that  $G_{tt} = 0$  which implies, by Einstein's equation (2.1), that this metric has a vanishing energy density,  $T_{tt}$ . However this is not a vacuum solution (which in 2+1 dimensions would necessarily be flat, Minkowski space-time) since the space-time has non-zero pressures, due to the non-vanishing of  $G_{rr}$  and  $G_{\theta\theta}$  for general  $A(r)$ . From a 3+1 perspective a source with no energy-density but a non-vanishing pressure is at the very least odd. However, we note in this regard that in the BTZ black hole solution, if one views the negative cosmological constant as a constant energy-momentum tensor  $-T_{\mu\nu}^{BTZ} \propto \Lambda g_{\mu\nu}$  with  $\Lambda < 0$  – then one has, not a zero energy density, but a negative energy density. As we will see, the reason for the vanishing of the energy-density of this solution comes from a cancellation of the kinetic and potential energies of the scalar field. This oddity points out the unusual and anti-intuitive (from a 3 + 1 dimensional perspective) relationship between sources and the gravitational field in 2 + 1 dimensions.

To this 2 + 1 space-time from eq. (2.2) we couple a real scalar field,  $\phi$ , with the Lagrangian

$$\mathcal{L}_S = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2\kappa} e^{-2\sqrt{\kappa}\phi} , \quad (2.4)$$

where the potential implied by eq. (2.4) is

$$V(\phi) = -\frac{1}{2\kappa} e^{-2\sqrt{\kappa}\phi}.$$

The presence of a potential which is a transcendental function can be compared to the use of the sinusoidal potential,  $V_{SG}(\phi) \propto \sin(k\phi)$ , in the sine-Gordon soliton [22]. In the context of dilatonic gravity in 2 + 1 dimensions such an exponential potential was used [23] to find a 2 + 1 dimensional, charged black hole. In addition such exponential potentials are widely used in cosmology in 3 + 1 in quintessence explanations of dark energy [24]. Such exponential potentials occur also generically for the dilaton field in string theory [3] and brane models. Also one might expect (and we will find) that a field in 2 + 1 dimensions would have logarithmic behavior for large  $r$ . Taking

the potential to be an exponential of the field will undo the logarithmic form of  $\phi$  and allow one to combine the potential term with the kinetic energy term  $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ .

From eq. (2.4) we need to calculate the energy-momentum tensor,  $T_{\mu\nu}$ , so that together with the Einstein tensor,  $G_{\mu\nu}$ , from eq. (2.3) we can set up the Einstein equation (2.1). Since we are assuming only  $r$  dependence of the metric we make the same assumption for the field,  $\phi$  i.e.  $\phi(t, r, \theta) \rightarrow \phi(r)$ . The general expression for  $T_{\mu\nu}$  associated with eq. (2.4) is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}_S. \quad (2.5)$$

Taking into account  $\phi(t, r, \theta) \rightarrow \phi(r)$  and the diagonal form of the metric (2.2) we get

$$T_{tt} = -A(r)\mathcal{L}_S \quad ; \quad T_{rr} = (\phi')^2 + \mathcal{L}_S \quad ; \quad T_{\theta\theta} = r^2\mathcal{L}_S. \quad (2.6)$$

Since  $G_{tt} = 0$  we need  $T_{tt} = 0$ . This implies that  $\mathcal{L}_S = 0$  for the condition that the solution satisfies  $\phi(t, r, \theta) \rightarrow \phi(r)$  i.e. we want

$$\mathcal{L}_S = 0 \rightarrow -\frac{1}{2}(\partial_r\phi)^2 + \frac{1}{2\kappa}e^{-2\sqrt{\kappa}\phi} = 0 \rightarrow \phi'(r) = \pm \frac{1}{\sqrt{\kappa}}e^{-\sqrt{\kappa}\phi}. \quad (2.7)$$

At the end we will need to verify that our solution,  $\phi(r)$ , does in fact satisfy  $\phi'(r) = \pm ge^{-\sqrt{\kappa}\phi}/\sqrt{\kappa}$ . Note that this condition also gives  $T_{rr} = (\phi')^2$  and  $T_{\theta\theta} = 0$ . Thus the only non-zero source term is the radial pressure,  $T_{rr}$ . The condition given in eq. (2.7) has the physical meaning that the kinetic energy of the scalar field (the derivative term) is balanced and canceled by the potential energy of the scalar field (the exponential term). This balancing of kinetic and potential energies of  $\phi$  makes it easy to understand the vanishing of the energy density,  $T_{tt}$ .

Putting the  $G_{\mu\nu}$ 's from eq. (2.3) together with the  $T_{\mu\nu}$ 's from eq. (2.5) we arrive at the system of equation we need to solve

$$\frac{A'(r)}{2rA(r)} = \kappa(\phi')^2 \quad ; \quad \frac{r^2}{4A^2(r)} \left[ -(A'(r))^2 + 2A(r)A''(r) \right] = 0. \quad (2.8)$$

In addition to finding an  $A(r)$  and  $\phi(r)$  which solve eq. (2.8) we also need  $\phi(r)$  to solve the Klein-Gordon equation coming from the Lagrange density eq. (2.4) and the metric eq. (2.2). In general, the Klein-Gordon equation for  $\phi$  in a curved background is

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) = -\frac{\partial V}{\partial\phi}, \quad (2.9)$$

where  $\sqrt{g} = \sqrt{\det(g_{\mu\nu})} = r\sqrt{A(r)}$ , and  $g$  is the determinant of the  $2 + 1$  dimensional metric. Taking into account that  $\phi(t, r, \theta) \rightarrow \phi(r)$  and  $V(\phi) = -\frac{1}{2\kappa}e^{-2\sqrt{\kappa}\phi}$  we find that the Klein-Gordon equation for  $\phi$  becomes

$$\phi''(r) + \left(\frac{1}{r} + \frac{A'(r)}{2A(r)}\right)\phi'(r) = \frac{1}{\sqrt{\kappa}}e^{-2\sqrt{\kappa}\phi}, \quad (2.10)$$

where the primes are differentiation with respect to  $r$ . It is straightforward to show that the system of Einstein equation eq. (2.8) plus Klein-Gordon equation eq. (2.10) are solved via

$$A(r) = Kr^2 \quad ; \quad \phi(r) = \frac{1}{\sqrt{\kappa}}\ln(r) . \quad (2.11)$$

Finally, recall that in order for this solution in eq. (2.11) to work it must also satisfy the condition  $\phi'(r) = \pm e^{-\sqrt{\kappa}\phi}/\sqrt{\kappa}$ . It is straightforward to check that eq. (2.11) does satisfy this condition. Note that since the only non-trivial metric component is,  $g_{00} = Kr^2$ , one could absorb the constant  $K$  into a redefinition of the time as  $t \rightarrow t'/\sqrt{K}$ . We keep the constant  $K$  for later comparison between the metric given by eq. (2.2) plus  $A(r) = Kr^2$  and the asymptotic form of the AdS metric as well as the metrics of reference [20].

### 3 Geometry of 2+1 dimensional spaces

In this section we will put the discussion into a more general context by considering a static, rotationally symmetric metric. It has the form

$$ds^2 = -a^2(r)dt^2 + b^2(r)dr^2 + r^2d\theta^2 \quad (3.1)$$

with  $\theta = 0$  and  $\theta = 2\pi$  being identified, if the circumferences of the circles  $r = \text{const.}$  are strictly monotonous functions of the radial coordinate. In this section, we take  $x^0 = t$ ,  $x^1 = r$ , and  $x^2 = \theta$ , and  $i, j = 0, 1, 2$ . We assume the functions  $a$  and  $b$  are positive, that  $r \geq 0$ , and that

$$\lim_{r \rightarrow 0} b(r) = 1 \quad (3.2)$$

to ensure that no conical singularity appears at  $r = 0$ , the center of symmetry.

The Ricci tensor,  $R_{uv}$ , for metric (3.1) in the coordinates chosen here, is diagonal. The three eigenvalues  $\lambda_0, \lambda_1, \lambda_2$  of  $R_{uv}$  (defined as  $R_{ii} = \lambda_i g_{ii}$ , where  $\lambda_i$  is the eigenvalue into the  $x^i$ -direction) are given by

$$\lambda_0 = -\frac{a''}{ab^2} + \frac{a'b'}{ab^3} - \frac{a'}{rab^2}, \quad (3.3)$$

$$\lambda_1 = -\frac{a''}{ab^2} + \frac{a'b'}{ab^3} + \frac{b'}{rb^3}, \quad (3.4)$$

$$\lambda_2 = \frac{b'}{rb^3} - \frac{a'}{rab^2}, \quad (3.5)$$

where the dashes denote  $d/dr$ . The curvature scalar,  $R$ , is given by

$$R = \lambda_0 + \lambda_1 + \lambda_2. \quad (3.6)$$

Let us look at the consequence if we require  $\lambda_0 = \lambda_1$ . From eqs. (3.3) and (3.4) this is equivalent to requiring  $a'/a = -b'/b$ , i.e., that the product  $a \cdot b$  is a positive constant. A coordinate transformation in metric (3.1) which consists in multiplying  $t$  by a positive constant can be compensated for by dividing  $a(r)$  by the same constant. Thus modulo this coordinate freedom one can say that the conditions  $a \cdot b = 1$  and  $\lambda_0 = \lambda_1$  are equivalent.

If  $\lambda_0 = \lambda_1$ , then  $\lambda_2$  simplifies from eq. (3.5) to

$$\lambda_2 = -\frac{2aa'}{r} = -\frac{1}{r} \cdot (a^2)'. \quad (3.7)$$

Assuming additionally  $\lambda_2 = 2c$  to be constant, we get from eq. (3.7) via  $2rc = -(a^2)'$  the solution for the metric components from (3.1)

$$a = \frac{1}{b} = \sqrt{\tilde{c} - cr^2} \quad (3.8)$$

with a further constant  $\tilde{c}$ . Inserting eq. (3.8) into eqs. (3.3) and (3.6) we get for this case  $\lambda_0 = 2c$  and  $R = 6c$ . Therefore from only the two conditions  $\lambda_0 = \lambda_1$  and  $\lambda_2 = \text{const.}$  we find that the space-time of (3.1) is one of constant curvature i.e.  $\lambda_0 = \lambda_1 = \lambda_2 = \text{const.}$

Eqs. (3.1) and (3.8) can be combined to give the metric

$$ds^2 = -(\tilde{c} - cr^2)dt^2 + \frac{dr^2}{\tilde{c} - cr^2} + r^2d\theta^2. \quad (3.9)$$

This space-time has a horizon at  $r = r_h$  where  $r_h$  is defined by  $\tilde{c} = cr_h^2$ . In contrast to the Schwarzschild metric in 3+1 dimensions the horizon for the space-time in (3.9) cannot be defined locally, since the character of the local isometry group of metric (3.9) does not change upon crossing the horizon (see the footnote 2). Furthermore the usual Schwarzschild solution in 3+1 dimensions is only static outside the horizon, whereas the metric in (3.9) is static everywhere. The metric in (3.9) has more in common with de Sitter



or Anti de Sitter space-time. Another feature of the metric in (3.9) is that the Ricci scalar is constant,  $R = 6c$ . Thus one is free to change the value of  $\tilde{c}$  by a suitable coordinate transformation. In order to satisfy eq. (3.2) one should set  $\tilde{c} = 1$ .

In case of  $c = 0$ , the metric (3.9) is completely flat. Nevertheless, a non-trivial feature can be constructed, if we change the character of the coordinate  $\theta$ : if  $\theta = 0$  and  $\theta = 2\pi\mu$  are identified with  $0 < \mu < 1$ , then we get a conical singularity at  $r = 0$  with a deficit angle of  $2\pi(1 - \mu)$ . Although the space-time is locally flat there are still global effects such as the deflection of null geodesics.

## 4 Conclusions and Summary

In this article we have found an  $r$ -symmetric, exact, analytical solution for a real scalar field coupled to gravity in  $2 + 1$  dimensions. The solution is given by the metric of the form (2.2) with the ansatz function  $A(r)$  and the scalar field,  $\phi(r)$  given in eq. (2.11). Such static solutions for gravitating real scalar fields in  $2 + 1$  dimensions have received very little attention outside of a few works [20]. The solution presented here can be considered a  $2 + 1$  dimensional version of the solutions in  $3 + 1$  dimensions of gravitating real scalar fields [15], [16], [17], and [18] but having features that are unique to  $2 + 1$  dimensions.

The  $\ln(r)$  behavior of the scalar field is what one expects for a massless field in  $2+1$  dimensions in the absence of gravity e.g. in  $2+1$  electrodynamics the scalar potential for a point charge goes as  $\ln(r)$ . Because of this one might think that the space-time given by eq. (2.2) with  $A(r) = Kr^2$  is flat. Calculating scalar quantities such as the Ricci scalar  $R_\alpha^\alpha = 2/r^2$  or the square of the Ricci tensor  $R^{\mu\nu}R_{\mu\nu} = 2/r^4$  one finds that these scalars are not zero for finite  $r$  as would be the case with Minkowski space-time. (In the limit  $r \rightarrow \infty$  these scalars do vanish).

In the spatial coordinates,  $(r, \theta)$  the metric is asymptotically flat. In other words as  $r \rightarrow \infty$  the spatial part of the metric eq. (2.2) approaches the 2D Euclidean line element  $dr^2 + r^2d\theta^2$ . The temporal part of the metric goes as  $Kr^2dt^2$  which is equivalent to the asymptotic behavior of the temporal part of Anti-de Sitter space-time  $\Lambda r^2dt^2$  where  $\Lambda$  is a positive cosmological constant. This feature of the metric – the spatial part of the metric being flat while the temporal part is equivalent to the asymptotic form of AdS space-time or

BTZ space-time – is connected with the unusual feature that the curvature of the space-time comes entirely from the radial pressure,  $T_{rr}$ . Additionally we can compare the metric of our solution  $ds^2 = Kr^2 dt^2 - dr^2 - r^2 d\theta^2$  with some of the static metric solutions discussed by E. Ayón-Beato et al. [20] who found metric solutions of the form

$$ds^2 = (r^2/l^2)dt^2 - (l^2/r^2)dr^2 - r^2 d\theta^2$$

where  $l$  is some length scale and  $1/l^2$  can be taken as a cosmological constant. In comparing our metric solution with this metric or with the asymptotic AdS metric, we see that the temporal parts have the same  $r^2$  behavior but the  $dr^2$  terms are different. Finally we note that our metric scales simply under the dilation of the  $r$ -coordinate  $r \rightarrow Cr'$  where  $C$  is some constant. Under this transformation the metric simply scales like  $C^2$  i.e.  $ds^2 \rightarrow C^2 ds'^2$ .

In closing we return to the odd energy-momentum tensor whose only non-vanishing component is  $T_{rr} = (\phi')^2$ . For the solution eq. (2.11) this gives

$$T_{rr} = \frac{1}{\kappa r^2} . \quad (4.1)$$

The components of the energy-momentum tensor (and by the Einstein equation eq. (2.1) the components of  $G_{\mu\nu}$ ) vanish like  $1/r^2$  at large  $r$ . Even though the energy-density of the source is zero (i.e  $T_{00} = 0$ ) one finds that the Komar mass [25] for this system gives physically reasonable results. The Komar mass is

$$m = \int_V (2T_{\mu\nu} - g_{\mu\nu}T) u^\mu \zeta^\nu dV , \quad (4.2)$$

where  $T = T^\mu_\mu$  is the trace of the energy-momentum tensor,  $u^\mu$  is a time-like vector,  $\zeta^\nu$ , is a Killing vector expressing the time translation invariance of the static metric eq. (2.2) and  $dV$  is the 2D volume element. For our solution, with  $T_{rr}$  and  $g_{\mu\nu}$  given by eq. (2.2) and eq. (2.11), we can write eq. (4.2) explicitly as

$$m = \int_V \sqrt{g_{00}}(T_{00} + T_{rr} + T_{\theta\theta})dV = \frac{2\pi}{\kappa} \int_0^\infty \frac{1}{r} dr = \frac{2\pi}{\kappa} [\ln(r)] \Big|_0^\infty . \quad (4.3)$$

This  $m$  will diverge due to both the singularity at  $r = 0$  and at  $r = \infty$ . This logarithmic divergence in the total energy is exactly the behavior one would expect for a massless scalar field in  $2 + 1$  in the absence of gravity.

As final note let us mention that quite recently, several papers appeared which are related to different other aspects of  $(2 + 1)$ -dimensional gravity: In

[26] and [27], massive gravity and supergravity is discussed; in [28], higher spin in topologically massive gravity is discussed; in [29] and [30] wormholes and star models are constructed in 2+1 dimensions; in [31], Birkhoff's theorem is generalized; in [32], [33], [34], [35], [36], [37], [38], and [39] various properties of the BTZ geometry are discussed.

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